S15 S2

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1. In a survey it is found that barn owls occur randomly at a rate of 9 per 1000 km^2.

(a) Find the probability that in a randomly selected area of 1000 km^2 there are at least 10 barn owls.

(2)

(b) Find the probability that in a randomly selected area of 200 km^2 there are exactly 2 barn owls.

(3)

(c) Using a suitable approximation, find the probability that in a randomly selected area of 50000 km^2 there are at least 470 barn owls.

(6)

a) x = # barn owls per 1000km^2 $\approx x \sim Po(9)$

$P(x \geq 10) = 1 - P(x \leq 9) = 1 - 0.5874 = 0.4126$

2

b) y = # barn owls per 200km^2 $\approx y \sim Po(1.8)$

$P(y=2) = \dfrac{e^{-1.8} \times 1.8^2}{2!} = 0.2678$

2

c) t = # barn owls per 50000km^2 $\approx t \sim Po(450)$

$\approx t \sim N(450, 450)$ $\quad P(t \geq 470)$ cc $P(t > 469.5)$

$t(t > 469.5)$

$\approx P\left(z > \dfrac{469.5 - 450}{\sqrt{450}}\right) = P(z > 0.92)$

$= 1 - \Phi(0.92) = 0.1788$

2

2. The proportion of houses in Radville which are unable to receive digital radio is 25%. In a survey of a random sample of 30 houses taken from Radville, the number, X, of houses which are unable to receive digital radio is recorded.

(a) Find $P(5 \leqslant X < 11)$

(3)

A radio company claims that a new transmitter set up in Radville will reduce the proportion of houses which are unable to receive digital radio. After the new transmitter has been set up, a random sample of 15 houses is taken, of which 1 house is unable to receive digital radio.

(b) Test, at the 10% level of significance, the radio company's claim. State your hypotheses clearly.

(5)

a) $X \sim B(30, 0.25)$ $\quad P(5 \leq X < 11) = P(X \leq 10) - P(X \leq 4)$

$= 0.8943 - 0.0979 \quad = \underline{0.7964}$

2

b) y = # houses that cannot receive signal

$y \sim B(15, 0.25)$ $\quad H_0: p = 0.25$
$\quad H_1: p < 0.25$

$P(y \leq 1) = 0.0802 \quad < 10\%$

$\therefore$ result is statistically significant
$\therefore$ enough evidence to reject null hypothesis
$\therefore$ Company's claim is supported

2

3. A random variable X has probability density function given by

$$f(x) = \begin{cases} kx^2 & 0 \leqslant x \leqslant 2 \\ k\left(1 - \frac{x}{6}\right) & 2 < x \leqslant 6 \\ 0 & \text{otherwise} \end{cases}$$

where k is a constant.

(a) Show that $k = \frac{1}{4}$ **(4)**

(b) Write down the mode of X. **(1)**

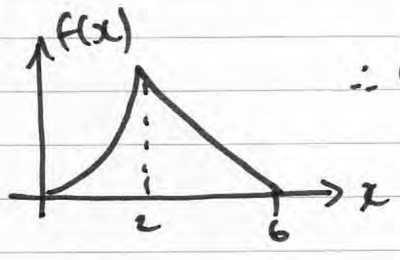
(c) Specify fully the cumulative distribution function $F(x)$. **(5)**

(d) Find the upper quartile of X. **(4)**

a) $\int f(x)dx = 1 \Rightarrow k\int_0^2 x^2 dx + k\int_2^6 1-\frac{x}{6}dx = 1$

$\Rightarrow k\left[[\frac{1}{3}x^3]_0^2 + [x-\frac{x^2}{12}]_2^6\right] = k\left[(\frac{8}{3}-0)+(6-3)-(2-\frac{4}{12})\right]=1$

$\Rightarrow k\left(\frac{8}{3}+3-\frac{5}{3}\right)=1 \Rightarrow 4k=1 \quad \therefore k=\frac{1}{4}$ #

b) 

$\therefore$ mode $= 2$

c) $0 \leq x \leq 2 \quad F(x) = \frac{1}{4}\int_0^x t^2 dt = \frac{1}{4}[\frac{1}{3}t^3]_0^x = \frac{1}{12}x^3$

$2 < x \leq 6 \quad F(x) = F(2) + \int_2^x \frac{1}{4}(1-\frac{t}{6})dt = \frac{1}{4}[t-\frac{t^2}{12}]_2^x + \frac{2}{3}$

$= \frac{1}{4}[(x-\frac{x^2}{12})-(2-\frac{4}{12})] + \frac{2}{3} = \frac{1}{4}x - \frac{x^2}{48} + \frac{1}{4}$

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{12}x^3 & 0 \le x \le 2 \\ \frac{1}{48}(12 + 12x - x^2) & 2 < x \le 6 \\ 1 & \text{otherwise} \end{cases}$$

d) $F(Q_3) = 0.75$

$\Rightarrow \frac{1}{48}(12 + 12x - x^2) = \frac{3}{4} \Rightarrow x^2 - 12x + 24 = 0$

$\Rightarrow (x-6)^2 = 36 - 24 \Rightarrow x - 6 = \pm\sqrt{12} = \pm 2\sqrt{3}$

$\therefore x = 6 - 2\sqrt{3} \quad Q_3 = 2.54$

2

4. The continuous random variable L represents the error, in metres, made when a machine cuts poles to a target length. The distribution of L is a continuous uniform distribution over the interval $[0, 0.5]$

(a) Find $P(L < 0.4)$.

(1)

(b) Write down $E(L)$.

(1)

(c) Calculate $Var(L)$.

(2)

A random sample of 30 poles cut by this machine is taken.

(d) Find the probability that fewer than 4 poles have an error of more than 0.4 metres from the target length.

(3)

When a new machine cuts poles to a target length, the error, X metres, is modelled by the cumulative distribution function $F(x)$ where

$$F(x) = \begin{cases} 0 & x < 0 \\ 4x - 4x^2 & 0 \leqslant x \leqslant 0.5 \\ 1 & \text{otherwise} \end{cases}$$

(e) Using this model, find $P(X > 0.4)$

(2)

A random sample of 100 poles cut by this new machine is taken.

(f) Using a suitable approximation, find the probability that at least 8 of these poles have an error of more than 0.4 metres.

(3)

a) $P(L<0.4) = \frac{4}{5}$

b) $E(L) = 0.25$

c) $V(L) = \frac{(b-a)^2}{12} = \frac{0.5^2}{12} = \frac{1}{48}$

d) $X \sim B(30, \frac{1}{5})$ X = Pole with an error > 0.4m

$P(X<4) = P(X \leq 3) = 0.1227$

e) $F(0.4) = 4(0.4) - 4(0.4)^2 = 0.96$

$\therefore P(X>0.4) = 0.04$

f) y = # Poles with an error > 0.4m

~~$y \sim B(100, 0.04)$~~

~~$\approx y \sim N(4, (\sqrt{3.84})^2)$~~ ~~$\mu = np = 4$~~

~~$\sigma^2 = np(1-p) = 4(0.96) = 3.84$~~

~~$P(y \geq 8)$ cc $P(y > 7.5)$~~

~~$P(y > 7)$~~

~~$\approx P\left(Z > \frac{7.5-4}{\sqrt{3.84}}\right)$~~

~~$P(Z > 1.78) = 1 - \Phi(1.78) = 0.0375$~~

alt $y \sim B(100, 0.04) \approx Po(4)$

$P(y \geq 8) = 1 - P(y \leq 7) = 0.0511$

5. *Liftsforall* claims that the lift they maintain in a block of flats breaks down at random at a mean rate of 4 times per month. To test this, the number of times the lift breaks down in a month is recorded.

(a) Using a 5% level of significance, find the critical region for a two-tailed test of the null hypothesis that 'the mean rate at which the lift breaks down is 4 times per month'. The probability of rejection in each of the tails should be as close to 2.5% as possible.
(3)

Over a randomly selected 1 month period the lift broke down 3 times.

(b) Test, at the 5% level of significance, whether *Liftsforall*'s claim is correct. State your hypotheses clearly.
(2)

(c) State the actual significance level of this test.
(1)

The residents in the block of flats have a maintenance contract with *Liftsforall*. The residents pay *Liftsforall* £500 for every quarter (3 months) in which there are at most 3 breakdowns. If there are 4 or more breakdowns in a quarter then the residents do not pay for that quarter.

Liftsforall installs a new lift in the block of flats.

Given that the new lift breaks down at a mean rate of 2 times per month,

(d) find the probability that the residents do not pay more than £500 to *Liftsforall* in the next year.
(6)

a) X = breakdowns per month $X \sim Po(4)$

$P(X \leq L) \simeq 0.025$
$P(X \leq 0) = 0.0183^*$
$P(X \leq 1) = 0.0916$
$\therefore L = 0$

$P(X \geq U) \simeq 0.025$
$1 - P(X \leq U-1) \simeq 0.025$
$P(X \leq U-1) \simeq 0.025$
$P(X \leq 7) = 0.9489$ $\therefore U-1 = 8$
$P(X \leq 8) = 0.9786^*$ $\therefore U = 9$

CR $(x=0) \cup (x \geq 9)$

b) $H_0 : \lambda = 4$
$H_1 : \lambda \neq 4$

$x = 3$ is NOT in the critical region
$\therefore$ result is NOT statistically significant
$\therefore$ not enough evidence to reject null
$\therefore$ Claim IS supported

c) ASL = $0.0183 + 0.0214 = 0.0397$ 3.97% cc

d) y = # breakdowns per 3 months $y \sim Po(6)$

do not pay if $y \geq 4 = 1 - P(y \leq 3) = 0.8488$

do not pay more than £500 in a year if

$y \geq 4$ occurs zero or once in 4 quarters

zero $\Rightarrow 0.8488^4 = 0.5191$
once $\Rightarrow 4 \times 0.8488^3 \times 0.1512 = 0.3699$ +

0.8890

blank

6. A continuous random variable X has probability density function $f(x)$ where

$$f(x) = \begin{cases} kx^n & 0 \leqslant x \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$$

where k and n are positive integers.

(a) Find k in terms of n.

(3)

(b) Find $E(X)$ in terms of n.

(3)

(c) Find $E(X^2)$ in terms of n.

(2)

Given that $n = 2$

(d) find $\text{Var}(3X)$.

(3)

a) $\int_0^1 f(x)dx = 1 \quad k\int_0^1 x^n\,dx = 1 \quad k\left[\frac{x^{n+1}}{n+1}\right]_0^1 = 1$

$= k\left(\frac{1}{n+1} - 0\right) = 1 \Rightarrow \frac{k}{n+1} = 1 \Rightarrow \underline{k = n+1}$

b) $E(X) = \int_0^1 x f(x)dx = \int_0^1 (n+1)x^{n+1}\,dx$

$= \left[\frac{n+1}{n+2}x^{n+2}\right]_0^1 = \frac{n+1}{n+2}$

c) $E(X^2) = \int_0^1 x^2 f(x)dx = \int_0^1 (n+1)x^{n+2}\,dx$

$= \left[\frac{n+1}{n+3}x^{n+3}\right]_0^1 = \frac{n+1}{n+3}$

d) $(n=2) \Rightarrow E(X) = \frac{3}{4} \quad E(X^2) = \frac{3}{5}$

$V(X) = E(X^2) - E(X)^2 = \frac{3}{5} - \frac{9}{16} = \frac{3}{80} \quad V(3X) = 9V(X) = \frac{27}{80}$

7. A bag contains a large number of 10p, 20p and 50p coins in the ratio $1:2:2$

A random sample of 3 coins is taken from the bag.

Find the sampling distribution of the median of these samples.

(7)

Median = 10p

$10,10,10 = \left(\frac{1}{5}\right)^3 = \frac{1}{125}$

$(\times 3)\ 10,10,20 = \left(\frac{1}{5}\right)^2 \times \frac{2}{5} \times 3 = \frac{6}{125}$

$(\times 3)\ 10,10,50 = \left(\frac{1}{5}\right)^2 \times \frac{2}{5} \times 3 = \frac{6}{125}$

$\frac{13}{125}$

Median = 20p

$(\times 3)\ 10,20,20 = \left(\frac{1}{5}\right) \times \left(\frac{2}{5}\right)^2 \times 3 \quad \frac{12}{125}$

$(\times 6)\ 10,20,50 = \left(\frac{1}{5}\right) \times \left(\frac{2}{5}\right)^2 \times 6 \quad \frac{24}{125}$

$20,20,20 = \left(\frac{2}{5}\right)^3 \quad \frac{8}{125}$

$(\times 3)\ 20,20,50 = \left(\frac{2}{5}\right)^3 \times 3 \quad \frac{24}{125} \Rightarrow \frac{68}{125}$

Median = 50p

$(\times 3)\ 10,50,50 \quad \frac{1}{5} \times \left(\frac{2}{5}\right)^2 \times 3 = \frac{12}{125}$

$(\times 3)\ 20,50,50 \quad \left(\frac{2}{5}\right)^3 \times 3 \quad \frac{24}{125} \quad \frac{44}{125}$

$50,50,50 \quad \left(\frac{2}{5}\right)^3 \quad \frac{8}{125}$

Median, m	10p	20p	50p
$P(M=m)$	$\frac{13}{125}$	$\frac{68}{125}$	$\frac{44}{125}$